Time dependence of expectation value $\frac{d\langle\hat{A}\rangle}{dt} = ?$

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

If time dependent:

$$\hat{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle \rightarrow |\frac{\partial\psi}{\partial t}\rangle = \frac{-i}{\hbar}\hat{H}|\psi\rangle$$

$$\langle\frac{\partial\psi}{\partial t}| = \frac{i}{\hbar}\langle\psi|\hat{H}$$

$$\frac{d\langle\hat{A}\rangle}{dt} = \frac{d}{dt}\langle\psi|\hat{A}|\psi\rangle$$

$$= \langle\frac{\partial\psi}{\partial t}|A|\psi\rangle + \langle\psi|\frac{\partial A}{\partial t}|\psi\rangle + \langle\psi|A|\frac{\partial\psi}{\partial t}\rangle$$

$$= \frac{i}{\hbar}\langle\psi|HA|\psi\rangle - \frac{i}{\hbar}\langle\psi|AH|\psi\rangle + \langle\psi|\frac{\partial A}{\partial t}|\psi\rangle$$

$$= \frac{i}{\hbar}\langle\psi|HA - AH|\psi\rangle + \langle\psi|\frac{\partial A}{\partial t}|\psi\rangle$$

$$= \frac{i}{\hbar}\langle\psi|[H, A]|\psi\rangle + \langle\frac{\partial A}{\partial t}\rangle$$

$$\frac{d}{dt}\langle\hat{A}\rangle = \frac{i}{\hbar}\langle[H, A]\rangle + \langle\frac{\partial A}{\partial t}\rangle$$

So if $\hat{A}$ is time independent:

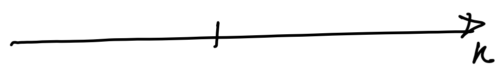
$$\frac{d}{dt}\langle \hat{A} \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{A}] \rangle$$

Easier to remember if write it in this form:

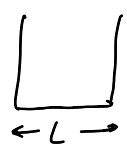
$$\boxed{[H, A] = -i\hbar \frac{dA}{dt}}$$

DENSITY OF STATES

Free electron $\quad E = \dfrac{\hbar^2 k^2}{2m}$ all k's are acceptable

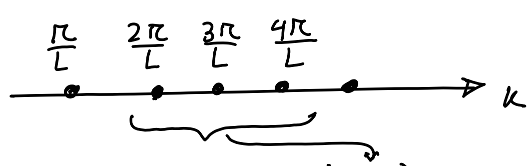


Electron in QW



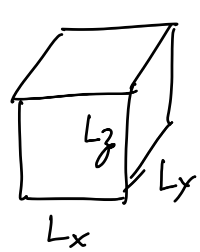
$E = \dfrac{\hbar^2 k_x^2}{2m}$ $\quad k_x$ are quantized:

$k_x = \dfrac{n\pi}{L} \quad n = 1, 2, \dots$



So for example, with this size of k, we only have three states for the electron.

Particle in a box

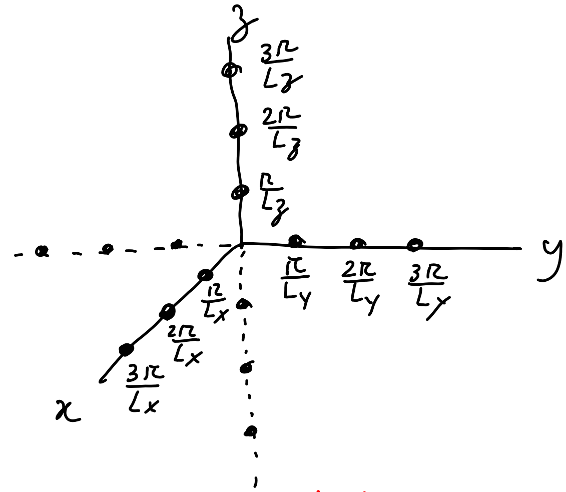


$$E = \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_y^2}{2m} + \frac{\hbar^2 k_z^2}{2m}$$

$$k_x = \frac{n_x \pi}{L_x} \qquad n_x = 1, 2, \ldots$$

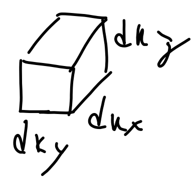
$$k_y = \frac{n_y \pi}{L_y} \qquad n_y = 1, 2, 3, \ldots$$

$$k_z = \frac{n_z \pi}{L_z} \qquad n_z = 1, 2, 3, \ldots$$



Question: How many states are available in a unit volume? Or: What is the density of states in k-space?

The volume is: $dk_x \, dk_y \, dk_z$



The volume for each state is: $\left(\frac{2\pi}{L_x}\right)\left(\frac{2\pi}{L_y}\right)\left(\frac{2\pi}{L_z}\right)$

So the number of states in volume dk is:

$$\frac{dk_x \, dk_y \, dk_z}{\frac{(2\pi)^3}{L_x L_y L_z}} = V \frac{dk_x \, dk_y \, dk_z}{(2\pi)^3}$$

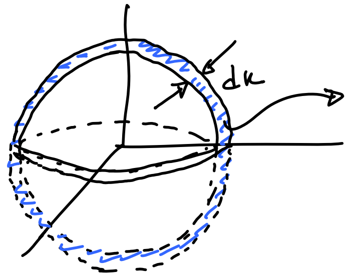
V: volume

So the # of states per unit volume is:

$$N(k) \, dk_x dk_y dk_z = \frac{dk_x \, dk_y \, dk_z}{(2\pi)^3} \times 2$$

spin up & down

In spherical coordinate: $dk_x dk_y dk_z = 4\pi k^2 dk$



$\rightarrow$ small volume $= 4\pi k^2 dk$

$$N(k)\,dk = \frac{1}{(2\pi)^3} 4\pi k^2 dk \times 2$$

$\uparrow$ spin

We usually like to know the density of states in Energy. $N(E) = ?$

$$N(E)\,dE = N(k)\frac{dk}{dE}dE$$

so we should calculate $\frac{dk}{dE}$:

But $E = \frac{\hbar^2 k^2}{2m} \Rightarrow dE = \frac{\hbar^2}{m} k\,dk \rightarrow \frac{dk}{dE} = \frac{m}{\hbar^2 k} \Rightarrow$

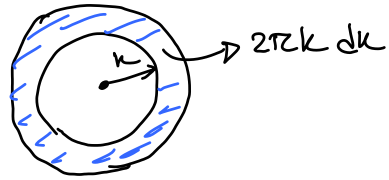
$$N(E)\,dE = \underset{\text{spin}}{2} \times \frac{4\pi k^2}{(2\pi)^3}\frac{m}{\hbar^2 k} dE$$

$$N(E) = \frac{2m}{2\pi^2\hbar^2}\frac{1}{k} = \frac{m}{\pi^2\hbar^2}\frac{\sqrt{2mE}}{\hbar}$$

$$\boxed{N^{3D}(E) = \frac{2^{1/2} m^{3/2}}{\pi^2\hbar^3}\sqrt{E}}$$

Energy DOS in 3D

In 2D:



$$N(E)\,dE = \overset{\text{spin}}{2}\;\frac{2\pi k\,dk}{(2\pi)^2}$$

$$N(E) = \frac{1}{\pi}\,k\,\frac{dk}{dE} = \frac{1}{\pi}\,k\,\frac{m}{\hbar^2 k} = \frac{m}{\pi\hbar^2}$$

$$\boxed{\overset{2D}{N}(E) = \frac{m}{\pi\hbar^2}}$$

In 1D

$$N(E)\,dE = \overset{\text{spin}}{2}\;\frac{dk}{2\pi}$$

$$N(E) = \frac{1}{\pi}\,\frac{dk}{dE} = \frac{1}{\pi}\,\frac{m}{\hbar^2 k} = \frac{m}{\pi\hbar^2}\,\frac{\hbar}{\sqrt{2mE}}$$

$$\boxed{\overset{1D}{N}(E) = \frac{m^{1/2}}{2^{1/2}\pi\hbar}\,\frac{1}{\sqrt{E}}}$$